TECHNICAL NOTE
D-10

ON SERIES EXPANSIONS IN MAGNETIC REYNOLDS NUMBER

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SUMMARY

Since exact flow solutions to magnetohydrodynamic flow problems are difficult to obtain, the majority of the analyses have been carried out on the assumption that the electrical conductivity (or magnetic Reynolds number) of the fluid is either small or very large. Effects in the intermediate range are generally not calculated even though they may at times be sizable. The essential feature of such an analysis is that the solution is assumed to be represented by a power series in the magnetic Reynolds number and only the leading term is considered. The present paper is a study of the nature of the higher order approximations. The method of obtaining the various terms is discussed. Several examples are then presented to illustrate the technique used and the character of the higher order solutions.

It is found that the higher approximations represent the flow field if they are found by iteration of the solution obtained by assuming that the magnetic Reynolds number is small or zero. Difficulties may be encountered, however, when the analysis is carried out by iteration on the solution obtained by assuming that the magnetic Reynolds number is infinite because the order of the differential equation is changed.

INTRODUCTION

When we are confronted with the task of solving a given magneto-fluid-dynamic problem, the question generally arises as to what assumptions to make in order to find a solution without an excessive amount of effort. Approximations may be made in either the fluid dynamic or the electromagnetic equations or both, or in the boundary conditions. One such simplification is to expand the velocity, pressure, magnetic field, etc., in powers of one or more parameters which are either very small or very large. The higher order effects can then be readily estimated or solved for in a logical manner if they are required. The series expansion in positive or negative powers of the so-called magnetic Reynolds number, $R_m = \sigma\mu U l$, (a list of symbols with definitions is given at the end of this section) has been used a number of times to solve a variety of problems. So far, only the zero order or the first term of the series in R_m or $1/R_m$ has been found, and the higher order effects are shown to be negligible or are not discussed. The purpose of this paper is to show how to obtain higher order approximations and to indicate the types of errors encountered when the series is cut off at the first term. Examples are then given to illustrate how the higher approximations affect the zero order solution. The geometry of the flow field is kept simple so that the influence of the approximation is not masked by the complexity of the flow field.

The stream is assumed to be laminar in all cases and its stability is not considered. The solutions that are given are then mathematically possible but may not be found experimentally. As is often the case in fluid dynamics, the flow may become turbulent so that more complicated phenomena will appear.

It is well known that the assumption of $R_m = 0$ is the same as assuming that the magnetic and electric fields induced by the fluid motion are negligible in comparison with the imposed magnetic field which generally has its source external to the flow field. The fluid-dynamic and electromagnetic equations can then be solved separately because, to this order, the magnetic field is assumed to be known throughout the flow field. The problem of finding a solution is therefore greatly simplified. When the magnetic Reynolds number is taken to be infinite the amount of simplification seems not to be so great. The two sets of equations separate only in special cases. The magnetic field configuration is not known at the outset because the lines of force move with the fluid. The fluid can flow along but not across the magnetic lines of force in this limit. Quite often then, the number of lines of force passing through a unit area is proportional to the density of the gas since the magnetic field is compressed with it. The magnetic field may be a function of time as well as the space coordinates and a steady configuration is achieved only when the streamlines and magnetic field lines become alined as time increases indefinitely.

Review of Literature

Before going into the theory of the development of the series expansion in $R_m = \sigma\mu U l$, it is desirable and informative to examine briefly the approach used to solve some typical magneto-fluid-dynamic problems. The number of problems which lend themselves to a solution without simplifying assumptions thus far are few. The solution found by Hartmann (ref. 1) for the steady flow of an incompressible electrical conducting fluid in a two-dimensional channel and the ingenious solution of Bleviss (ref. 2) for the flow of air as a real gas between parallel plates in relative motion are two examples of complete solutions. The first (ref. 1) represents the flow of mercury, salt water, liquid sodium, or any other liquid conductor through a two-dimensional channel under the action of electric and magnetic fields. The second is the Couette flow solution for air between parallel plates and considers the compressibility, variation in the specific heats, and conductivity with the temperature of the air. At the present time, it represents the most complete solution to a magnetogasdynamic problem.

The incompressible and viscous flow solution at a two-dimensional stagnation point which was found by Neuringer and McIlroy (ref. 3) for a magnetic field perpendicular to the surface may also be classed as an exact solution with certain reservations. The complete electromagnetic and fluid-dynamic equations were solved numerically but the induced magnetic field at the wall was assumed to be zero. This corresponds to the assumption that an image of the flow field exists on the opposite side of the wall and not that the wall is a perfect conductor so that the electric potential in the wall is zero. This example illustrates a difficulty which arises in finding the initial conditions which are to be used to solve the flow field numerically. The induced magnetic field at a boundary often cannot be found until the entire flow field is known. Therefore, step-by-step integration is not expected to be a promising method of analysis unless some sort of an iteration process is also used in conjunction with it. The analysis in reference 2 assumed that the flow field was like an infinitely long solenoid in which the induced field at the outer wall is known to be zero. The magnetic field at the outer wall was then known independently of and without prior knowledge of the flow field between the walls. Naturally then, a key to finding complete solutions lies in choosing simple configurations with boundaries designed so that most or all quantities are known a priori.

$\underline{R_m \to 0}$.- The problems which have been solved on the assumption that the induced magnetic field is negligible in comparison with the imposed magnetic field are reported in references 4 to 14. This approximation has been in use for a number of years to compute the torque, electric currents, etc., in electric motors and generators. The boundary-layer problems reported in references 4 to 7 (Kemp, Rossow, Lykoudis, and Leadon, respectively) were solved without expanding in another parameter. In analyzing the flow in pipes (refs. 8 to 10) Shercliff assumes that the parameter $4\pi\sigma\mu/\nu$ is nearly zero which, as he points out, is equivalent to neglecting the induced field. The solutions reported in the references 10 to 14 (Shercliff, Rossow, Kemp and Petschek, deLeeuw, and Chester, respectively) were solved by first assuming that the induced magnetic field is small and then expanding in a series in $Q = \sigma B^2 l/\rho U$ or $M = \sqrt{\sigma/\eta}\,Bl$. Such a double series expansion is a powerful method for solving magneto-fluid-dynamic problems. The zero and first order terms were found for the solutions reported in references 12 (Kemp and Petschek) and 13 (deLeeuw). Several higher order terms were found for the solutions reported in references 10 (Shercliff), 11 (Rossow), and 14 (Chester). The variety of problems solved by this technique covers the flow at a stagnation point (refs. 4 and 5), the flow in the boundary layer on a flat plate or on a wedge (refs. 6, 7, and 11), the flow in pipes (refs. 8, 9, and 10), Stokes flow about a sphere moving in the direction of the magnetic lines of force (ref. 14), the interaction of a shock wave and a magnetic field (ref. 13), and the flow through an elliptical solenoid (ref. 12).

$\underline{R_m = \infty}$.- When any of the quantities such as the characteristic dimensions, velocity, or electrical conductivity of the flow field being studied are very large (e.g., in stars), the magnetic Reynolds number $R_m = \sigma\mu Ul$ may become many orders of magnitude greater than unity. The assumption that terms involving $1/R_m$ or the magnetic diffusivity $1/\sigma\mu$ are then negligible is reasonable. The approximation is used in references 15 to 33 to estimate astrophysical phenomena and to solve various magneto-fluid-dynamic problems. A discussion and review of papers dealing with flows of this type is given by Elsasser in reference 34. The material here is intended as an extension of that work.

Some of the earliest work along these lines appears to have been carried out by Cowling (ref. 15). He postulates a possible process for the origin of the magnetic fields in sunspots. It is assumed that the lines of force diffuse through the fluid at a rate which is slow compared to the fluid velocity in a sunspot column. The magnetic field of the sun is then essentially swept along with the fluid and projected out through the surface of the sun with the fluid. This model is justified by an order analysis without specifically mentioning the magnetic Reynolds number. Recently, a device has been built by Patrick (ref. 16) which operates in reverse to the one just described. Instead of driving a magnetic field with a fluid, a gas column is accelerated by a strong magnetic field in a copper-shielded tube. The time duration of the cycle is so short and the velocity of the gas so high that the magnetic field cannot diffuse through the material. The magnetic Reynolds number is effectively infinite.

The waves which propagate along the magnetic lines of force when the conductivity is infinite and the viscosity is zero (Alfvén waves) have been studied by Alfvén and Walén (see, e.g., refs. 17 and 18). Since the fluid has mass and the magnetic lines are under tension, a wave or waves can propagate along the lines of force in the same manner that a wave travels along a heavy rope or wire. The electrical conductivity of the fluid must be high enough so that the fluid and magnetic field may be considered attached to each other (i.e., $1/\sigma\mu \ll 1$).[1] It was also noted by Batchelor (ref. 19) that a magnetic field cannot arise spontaneously in a medium due to turbulence unless $\sigma\mu > 1$.

The recent widespread use of the infinite conductivity concept may be attributed partly to its convenience and partly to the fact that it is a reasonable assumption for stars and for masses of highly conducting and rapidly moving gases. Although the physical dimensions of an electromagnetic pinch are not large, the times involved are small and the velocity of the compression wave and the electrical conductivity are high enough to make the magnetic Reynolds number large. From this point of view the concept of the "snowplow shock wave" is introduced and analyzed by

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[1]The concept of the magnetic field lines being frozen into the fluid when the magnetic Reynolds number is infinite is generally attributed to Alfvén (see, e.g., ref. 18).

Rosenbluth, et al., in reference 20. The structure and characteristics of weak waves and of shock waves in a highly conducting medium in the presence of a magnetic field are treated by Kaplan and Stanyukovich, Helfer, Sen, Cabannes, Burgers, and Pai in references 21 to 26, respectively. The resolution of an initial shear flow discontinuity in a compressible, perfectly conducting medium is treated by Bazer (ref. 27). Stewartson (ref. 28) used the large conductivity approximation to analyze the destabilization effect of a magnetic field observed experimentally by Lehnert (ref. 29) for a shearing motion in mercury. Longhead (ref. 30) sets up a finite difference method for solving problems when the conductivity is infinite.

The term in the electromagnetic equations which is dropped because $1/R_m$ is negligible contains the highest derivative in the equation. A singular perturbation technique must then be used to find higher approximations in much the same manner employed in problems characterized by a high viscous Reynolds number. This characteristic of the electromagnetic equations was recognized and used by Michael (ref. 31) to find the incompressible flow around a cylinder moving through a transverse magnetic field. The flow far from the body ($R_m = \infty$) is matched with the magnetic boundary-layer flow ($R_m \neq \infty$) near the cylinder surface.

The characteristics of a given configuration can be bracketed quite well when an analysis is made at $R_m = 0$ and ∞, even though the solution for an arbitrary value of R_m cannot be found. This was done by Marshall (ref. 32) to study the structure of a magnetohydrodynamic shock wave and by Hide (ref. 33) to study the stability and wave growth of a liquid with the more dense fluid initially at the top.

Other series expansions.- The references cited so far comprise part of a group of solutions found when it is assumed that the magnetic Reynolds number is either very large or very small. Another group of solutions has been obtained by series expansions in other quantities. The approximation that the stream velocity is small or large was used by Elsasser (refs. 35 and 36) and by Cowling and Hare (ref. 37) to study the decay of magnetic fields in large bodies such as the core of the earth or in stars. In this work, Elsasser introduced the exponential decay law $\vec{A} = \vec{A}_0 e^{-k^2(t/\sigma\mu)}$ where the values of k are determined as the eigenvalues of the equation $\nabla^2\vec{A}_0 + k^2\vec{A}_0 = 0$. Lundquist (ref. 38) expanded the magnetic potential vector $\vec{A}$ (where, curl $\vec{A} = \vec{H}$) in a power series in time, $\vec{A} = \sum_n (\vec{A}_n t^n/n!)$, to investigate the rate of distortion of circular lines of force to an elliptical shape by the converging flow of an electrically conducting fluid. The flow around a sphere moving through a strong magnetic field was found by Stewartson (ref. 39). He assumes that the velocity perturbations are small and then investigates the motion at time $t = 0$ when the sphere has just started moving and the time $t = \infty$ when a steady state is present.

PRINCIPAL SYMBOLS

b,B	magnetic induction
E	electric field intensity
H	magnetic intensity, $\frac{B}{\mu}$
J	electric current density
l	characteristic length
M	Hartmann number, $\sqrt{\frac{\sigma}{\eta}}\,Bl = \sqrt{ReR_mR_h}$
p	pressure
Q	magnetic parameter, $\frac{\sigma B^2 l}{\rho U} = R_mR_h$
Re	viscous Reynolds number, $\frac{Ul}{\nu}$
R_h	magnetic pressure number, $\frac{B^2}{\mu\rho U^2}$
R_m	magnetic Reynolds number, $\sigma\mu Ul$
s	Laplace transform variable
t	time
u,v,w	velocity components in x,y,z directions
U	velocity
x,y,z	coordinate axes, x alined with free stream
δ	half width of two-dimensional channel
θ	excess charge density
λ	magnetic viscosity, $\frac{1}{\sigma\mu}$
η	coefficient of viscosity
μ	magnetic permeability

μ_o magnetic permeability of free space

ν kinematic viscosity, $\frac{\eta}{\rho}$

ρ density of fluid

ρ_o density of air at sea level

σ electrical conductivity

Subscripts

o basic quantity

∞ free stream

x,y,z component along coordinate axes

Superscripts

$(\vec{\ })$ vector quantity

$(\bar{\ })$ Laplace transform of the quantity

The units and conversion constants to be used in the evaluation of various parameters are listed in the appendix.

SERIES DEVELOPMENT

It is necessary to define the kind of flow fields which are to be dealt with before the series expansions in magnetic Reynolds number can be discussed. It is assumed that Maxwell's equations may be used in the simplified form

$$\vec{\nabla}\cdot\vec{B} = 0 , \qquad \vec{\nabla}\cdot\vec{E} = 0 \tag{1}$$

$$\text{curl } \vec{H} = \vec{J} \tag{2}$$

$$\text{curl } \vec{E} = -\frac{\partial\vec{B}}{\partial t} \tag{3}$$

The symbol $\vec{B}$ denotes the magnetic induction, $\vec{E}$ denotes the electric field intensity, and $\vec{J}$ denotes the electric current density with $\mu\vec{H} = \vec{B}$, and $\epsilon\vec{E} = \vec{D}$ where μ is the magnetic permeability and ϵ the permittivity of the fluid. At the interface between two media, the tangential component of $\vec{E}$ and the normal component of $\vec{B}$ are assumed to be continuous. The flow field is assumed to be homogeneous so that the magnetic permeability μ and permittivity ϵ are constant. The displacement and Hall currents are considered to be negligible in comparison with the quantities which are retained. Ohm's law for a moving fluid will be written in its simplified form as

$$\vec{J} = \sigma(\vec{E} + \vec{U}\times\vec{B}) \tag{4}$$

The density of the fluid is assumed to be high enough so that the mean free path is very small compared with the characteristic dimensions of the flow field. The analysis will then treat the fluid as a continuum and not as a group of individual particles. In the illustrative examples which are presented the density, electrical conductivity, and fluid properties are taken as constant. The charge density θ will be assumed to be zero. The equation of motion of the fluid may then be written as

$$\rho\,\frac{D\vec{U}}{Dt} + \vec{\nabla}p = \vec{J}\times\vec{B} + \eta\nabla^2\vec{U} \tag{5a}$$

$$= \sigma(\vec{E} + \vec{U}\times\vec{B})\times\vec{B} + \eta\nabla^2\vec{U} \tag{5b}$$

$$= \frac{1}{\mu}\,(\text{curl }\vec{B})\times\vec{B} + \eta\nabla^2\vec{U} \tag{5c}$$

and the continuity equation is

$$\vec{\nabla}\cdot\vec{U} = 0 \tag{5d}$$

The dimensionless quantities $\tau = tU_o/l$, $\tilde{p} = p/\rho U_o^2$, $\chi = x/l$, $\xi = y/l$, $\zeta = z/l$, $\vec{B} = B_o\vec{\beta}$, $\vec{U} = U_o\vec{U}$, and $\vec{E} = E_o\vec{E}$ may be introduced into equations (5b) and (5c) to yield

$$\frac{D\vec{U}}{D\tau} + \vec{\nabla}\tilde{p} = \frac{\sigma E_o l}{\rho U_o^2}\,\vec{E} + Q(\vec{U}\times\vec{\beta})\times\vec{\beta} + \frac{\nabla^2\vec{U}}{Re} \tag{5e}$$

$$\frac{D\vec{U}}{D\tau} + \vec{\nabla}\tilde{p} = R_h(\text{curl }\vec{\beta})\times\vec{\beta} + \frac{\nabla^2\vec{U}}{Re} \tag{5f}$$

where the magnetic parameter $Q = \sigma B_o^2 l/\rho U_o$ is related to the magnetic pressure number $R_h = B_o^2/\mu\rho U_o^2$ and the magnetic Reynolds number $R_m = \sigma\mu U_o l$ by $Q = R_h R_m$. The quantity R_h is essentially independent of the magnetic Reynolds number R_m and may be quite sizable even though R_m is small. Physical situations can then exist in which the product

$Q = R_h R_m$ may be of the order of one even though R_m is small. The series expansion being studied is made with respect to R_m instead of R_h because, as will be seen in a later section, R_m controls the magnitude of the induced magnetic field. A suitable modification of the equations of motion would remove the constant density restriction without requiring a change in the equations for the magnetic and electrical fields.

The series expansion techniques to be discussed will apply to other groups of problems but, in order to achieve simplicity, the discussion will be limited to problems encompassed by the foregoing description.

Equation of Motion for Magnetic Field

The electric field intensity $\vec{E}$ and current density $\vec{J}$ can be eliminated from Ohm's law by taking the curl of equation (4) and substituting the equations (1), (2), and (3).

$$\text{curl } \vec{J} = \sigma\vec{\nabla} \times (\vec{E} + \vec{U} \times \vec{B})$$

When equations (2) and (3) are introduced, the result is,

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \sigma\mu\left[-\frac{\partial\vec{B}}{\partial t} + \vec{\nabla} \times (\vec{U} \times \vec{B})\right]$$

or from equation (1) and vector relations,

$$\nabla^2\vec{B} = \sigma\mu\left[\frac{\partial\vec{B}}{\partial t} - \vec{\nabla} \times (\vec{U} \times \vec{B})\right] \tag{6a}$$

or

$$\frac{\partial\vec{B}}{\partial t} = \vec{\nabla} \times (\vec{U} \times \vec{B}) + \frac{1}{\sigma\mu}\nabla^2\vec{B} \tag{6b}$$

Equations (6) relate the changes in the magnetic field to the motion of the fluid and will therefore be referred to as the equation of motion of the magnetic field.

If dimensionless quantities are introduced into equation (6a), the result is

$$\nabla^2\vec{\beta} = \sigma\mu U_o l\left[\frac{\partial\vec{\beta}}{\partial\tau} - \vec{\nabla} \times (\vec{U} \times \vec{\beta})\right] \tag{7}$$

The parameter $\sigma\mu U_o l$ is well known as the magnetic Reynolds number which was mentioned previously and is denoted in this paper by the symbol R_m. The quantity $1/\sigma\mu$ is the magnetic diffusivity. It is sometimes called the magnetic viscosity because of the similarity of the roles played by it and the kinematic viscosity, ν.

A number of the references cited in the introduction discuss situations in which the magnetic Reynolds number is very large or very small. With a few exceptions, it may be said that man-made devices have a small value of R_m and stellar devices such as the sun, stars, and core of the earth have large values for the magnetic Reynolds number.

Magnetic Reynolds Number Approximations

Solutions to equations (5) and (6) can in certain cases be found without making further simplifying assumptions. In the majority of cases however, it is necessary to make approximations of one sort or another. The equations (5) and (6) can be simplified a great deal by setting R_m equal to either zero or infinity. Higher approximations would then be obtained by iteration on this flow field. The magnitude of the iterated quantity would be of the order of R_m or $1/R_m$ depending on how the basic solution was obtained. One would then think of the process as being one in which the various functions are expanded in a power series in R_m or $1/R_m$.

The foregoing ideas are not new and have been discussed in a number of papers. The purpose of the present paper is to enlarge upon the manner in which higher approximations are obtained and to illustrate their effect on the characteristics of several simple flow fields. The method of approach to the series expansion will be treated in this section.

First method for small magnetic Reynolds number.- Consider first the form of equations (5b) and (6a) when the magnetic Reynolds number is very small. If R_m (or σ) is exactly zero, the differential equation for the fluid motion reduces to its nonmagnetic form and $\nabla^2\vec{B} = 0$. The next step is to expand the various parameters in a series in positive powers of R_m.

$$\left.\begin{aligned}
\vec{U} &= \vec{U}_o + \vec{U}_1 R_m + \vec{U}_2 R_m^2 + \ldots \\
\vec{B} &= \vec{B}_o + \vec{B}_1 R_m + \vec{B}_2 R_m^2 + \ldots \\
p &= p_o + p_1 R_m + p_2 R_m^2 + \ldots \\
&\text{etc.}
\end{aligned}\right\} \tag{8}$$

where U_o, U_1, B_o, B_1, p_o, p_1, etc., are functions of the space coordinates and time. The zero order solution (subscript zero) is the nonmagnetic flow field. The magnetic field B which is imposed by an external source is not distorted or changed by the fluid motion because to this approximation there is no interaction between it and the fluid. Higher approximations are obtained by iteration on the nonmagnetic flow field. The differential equations are obtained by substituting the series expressions (8) into the differential equations (5) and (6) and then equating the terms which contain like powers of R_m.

Second method for small magnetic Reynolds number.- Another approach may be used when the magnetic Reynolds number is small. Assume that the induced magnetic field is negligible in comparison with the imposed magnetic field but that it is not zero. Once again equation (6a) reduces to $\nabla^2\vec{B} = 0$. The equation of motion for the fluid does not reduce to the nonmagnetic form however, because the force term $\sigma(\vec{E}+\vec{U}\times\vec{B})\times\vec{B}$ in equations (5) does not vanish. In other words, the parameter $R_m = \sigma\mu U l$ is assumed to be negligibly small but the quantity $Q = \sigma B^2 l/\rho U = R_m R_h$ is not small. Such a situation is possible when strong magnetic fields are imposed on the flow field. A large simplification is achieved because the magnetic field strength in the force term in the momentum equations may be taken as the externally imposed magnetic field which is known at the outset. The analysis of the fluid and magnetic field motion is disconnected by this process, and each is found as a separate quantity from the previous approximation. The analysis is carried out in the following way. The quantity $\vec{B}_o$ is defined as the imposed magnetic induction and $\vec{b}$ as the induced magnetic induction arising from electric currents flowing in the fluid. The complete magnetic field is given by the sum

$$\vec{B} = \vec{B}_o + \vec{b} \tag{9}$$

Since the quantity $\vec{b}$ arises from electric currents flowing in the fluid conductor, its magnitude can be found from relations given in texts on electricity and magnetism (see, e.g., Stratton, ref. 40).

A magnetic vector potential $\vec{A}$ is defined with the properties,

$$\left.\begin{aligned} \vec{\nabla}\times\vec{A} &= \vec{b} \\ \vec{\nabla}\cdot\vec{A} &= 0 \end{aligned}\right\} \tag{10}$$

Its magnitude is then given by

$$\vec{A} = \frac{1}{4\pi}\iiint\limits_V \frac{\mu\vec{J}\,dv}{r}$$

where V is the volume in which electric currents are flowing and r is $\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$, the distance between the point being considered and the location of the electric current element. When the electric current density $\vec{J}$, given by equation (4), is substituted into the above expression, the magnetic induction brought about by electric currents flowing in the fluid is given by

$$\vec{b} = \vec{\nabla}\times\vec{A} = \frac{\sigma\mu}{4\pi}\iiint_V \vec{\nabla}\times\frac{[\vec{E}+\vec{U}\times(\vec{B}_o+\vec{b})]}{r}\,dv \tag{11}$$

If the quantities in equation (11) are put in dimensionless form, the induced magnetic field strength $\vec{b}$ is seen to be proportional to R_m. The dimensionless parameter E_o/U_oB_o, which also appears in equation (11), expresses the relative magnitudes of the imposed and induced potentials (see appendix). The integral equation can then be solved by successive approximations with the first one consisting of the assumption that $\vec{B}_o \gg \vec{b}$ so that

$$\vec{b}_1 = \frac{\sigma\mu}{4\pi}\iiint_V \nabla\times\frac{(E_o+\vec{U}_o\times\vec{B}_o)}{r}\,dv \tag{12}$$

The next approximation $\vec{b}_2$ is obtained by substituting $(\vec{B}_o+\vec{b}_1)$ for $\vec{B}_o$ and $(\vec{U}_o+R_m\vec{U}_1)$ for $\vec{U}_o$ in equation (12). It will be found to be proportional to R_m^2. If the iteration scheme is followed, a series expansion in R_m is obtained for $\vec{b}$. The equation of motion for the fluid is treated the same as it was in the first method with one exception; that is, the first approximation to the solution (subscript zero) is obtained with the electromagnetic force term $\sigma(\vec{E}+\vec{U}\times\vec{B})\times\vec{B}$ included and not set equal to zero. The same series expressions (8) are used by both methods to find the differential equations for the various iterations.

Large magnetic Reynolds number.- When R_m approaches infinity, equation (6b) reduces to

$$\frac{\partial\vec{B}}{\partial t} = \vec{\nabla}\times(\vec{U}\times\vec{B}) \tag{13}$$

The magnetic field is then characterized by the fact that it is influenced strongly by the motion of the fluid. As pointed out previously, Alfvén has called the process one in which the magnetic lines of force can be regarded as frozen into the fluid.

The equation of motion for the fluid is used in the form given by equation (5c), so that no difficulty arises because of the large size of R_m. Successive approximations in the example to be studied will be found by formal expansion of the various physical quantities in inverse powers of R_m.

$$\left.\begin{aligned} \vec{U} &= \vec{U}_o + \frac{\vec{U}_1}{R_m} + \frac{\vec{U}_2}{R_m^2} + \ldots \\ \vec{B} &= \vec{B}_o + \frac{\vec{B}_1}{R_m} + \frac{\vec{B}_2}{R_m^2} + \ldots \\ p &= p_o + \frac{p_1}{R_m} + \frac{p_2}{R_m^2} + \ldots \\ &\text{etc.} \end{aligned}\right\} \quad (14)$$

The order of the differential equation (6b) for the magnetic field is reduced when the term $(1/\sigma\mu)\nabla^2\vec{B}$ is discarded as a first approximation. The resulting singular perturbation nature of the analysis causes difficulty in iterating on the basic solution to find higher approximations. Special care must be exercised to ensure that the correct solution is obtained, and one should be wary of the results obtained by the straightforward formal iteration method discussed here. The examples illustrate some of the characteristics to be expected from the solution of problems found by this technique.

Small Magnetic Reynolds Number Examples

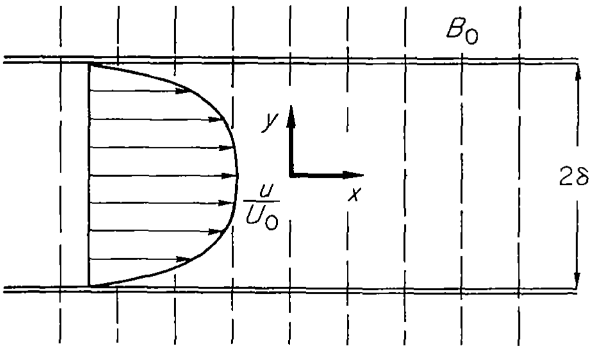
Several examples are presented in this section to illustrate the technique to be used in finding the solution to problems by the small magnetic Reynolds number methods. It would be desirable to find the higher approximations to a solution which is published - for instance, that of Kemp and Petschek in reference 12 or Rossow in reference 5 or 11. The basic solutions are so complicated, however, that the mathematical details would be difficult and would mask the effects of the higher approximations thereby defeating the intended purpose of the example. Instead of working with such complicated flow fields, several simple problems will be considered. They possess small practical value but have the virtue of being simple enough to be readily amenable to analysis and to the observation of higher order effects.

The following examples will be treated:

(1) Hartmann type channel flow
  (a) Analysis by first method
  (b) Analysis by second method

(2) Inviscid channel flow with finite transverse magnetic field
  (a) Channel of finite height
  (b) Channel of infinite height

Hartmann type channel flow.- The two-dimensional flow of viscous, incompressible, electrically conducting fluid through an infinitely long channel (sketch (a)) was first analyzed by Hartmann (ref. 1) and the solution is reproduced in a number of papers. The differential equations and boundary conditions are simple enough that the problem can be worked out in closed form without further simplification. The solution is repeated here to serve as a test for the approximate methods.



Sketch (a)

When the fluid flows through an annulus formed by two concentric cylinders, the electric current lines in the fluid are closed loops (circles) around the inner cylinder. The electric field $\vec{E}$ may then be set equal to zero because the magnetic field is not in motion. If the radial depth of the annulus is small compared with the diameter, the flow may be assumed to be two-dimensional and the curvature of the flow field in the z direction neglected. The imposed magnetic field is directed radially outward (+y) and is therefore perpendicular to the stream direction. The pole faces of the magnet are stationary and are assumed to be perfect conductors so that $\vec{E} = 0$ in them. The flow field is assumed to be so long that the effect of the end of the channel is negligible and the streamwise variation in the various quantities is zero. Under these circumstances the closed form of the equations for the velocity, magnetic field, and pressure are given by the expressions similar to those developed by Hartmann (ref. 1). The expressions differ because the potential $\vec{E}$ is assumed to be zero here.

$$u = U_0 \frac{\cosh M - \cosh M(y/\delta)}{\cosh M - 1} \tag{15a}$$

$$B_y = B_0 \tag{15b}$$

$$B_x = B_0 R_m \frac{(y/\delta)\cosh M - (1/M)\sinh M(y/\delta)}{1 - \cosh M} \tag{15c}$$

$$\frac{\partial p}{\partial x} = \frac{\eta U_0 M^2}{\delta^2} \frac{\cosh M}{1 - \cosh M} \tag{15d}$$

where M is the Hartmann parameter, $\sqrt{\sigma/\eta}\, B_0\delta$. The combined magnetic and fluid pressure does not change across the channel.

$$\frac{\partial}{\partial y}\left(p + \frac{B_x^2}{2\mu}\right) = 0$$

The pressure is then given by

$$p_o - P = (x - x_o)\,\frac{\eta U_o M^2}{\delta^2}\,\frac{\cosh M}{1 - \cosh M} + \frac{{B_o}^2 {R_m}^2}{M^2}\left[\frac{y(M/\delta)\cosh M - \sinh M(y/\delta)}{1 - \cosh M}\right]^2 \tag{15e}$$

where P is the reference pressure at some station $x = x_o$ and $y = 0$. The results given by equations (15) will now be found by the first and second methods outlined in the last section.

First method.- The differential equations for the channel flow problem described in the previous paragraph are found from equations (1) through (6).

$$\frac{\partial B_y}{\partial y} = 0 \tag{16a}$$

$$\frac{\partial B_x}{\partial y} = -\mu J_z = -\sigma\mu u B_y \tag{16b}$$

$$= -R_m B_o \frac{u}{U_o \delta}$$

$$\frac{\partial p}{\partial x} = \eta\,\frac{\partial^2 u}{\partial y^2} - R_m\,\frac{{B_o}^2}{\delta\mu U_o}\,u \tag{16c}$$

$$\frac{\partial p}{\partial y} = -\,\frac{1}{\mu}\,B_x\,\frac{\partial B_x}{\partial y} \tag{16d}$$

$$\frac{\partial^2 B_x}{\partial y^2} = -\sigma\mu B_o\,\frac{\partial u}{\partial y} \tag{16e}$$

The velocity components in the y and z directions are zero everywhere. Equation (16e) is equivalent to equation (16b) since the flow is assumed to be steady. The boundary conditions are

$$\begin{array}{ll} u = 0 & \text{at } y = \pm\delta \\ u = U_o & \text{at } y = 0 \\ p = P & \text{at } x = x_o\,,\ y = 0 \\ B_x = 0 & \text{at } y = 0 \\ E = 0 & \text{everywhere} \end{array}$$

The velocity, pressure, and magnetic field are assumed to be represented by the series expressions

$$\left.\begin{aligned} u &= u_o + u_1 R_m + u_2 R_m^2 + \ldots \\ p &= p_o + p_1 R_m + p_2 R_m^2 + \ldots \\ B_x &= B_{x_o} + B_{x_1} R_m + B_{x_2} R_m^2 + \ldots \\ B_y &= B_{y_1} + B_{y_1} R_m + B_{y_2} R_m^2 + \ldots \end{aligned}\right\} \tag{17}$$

It is noted that the equations (15) for the various quantities in the flow field are a function of the Hartmann parameter M which contains the magnetic Reynolds number through the relationship $M^2 = (B^2 l/\mu\eta U_o)R_m$. For this particular problem then, an expansion in M would be more suitable. The series (17), however, is a more general representation which may be applied to more complicated problems. As mentioned previously, the solution to the channel flow problem by the series (17) is carried out here for the purpose of gaining an understanding of this expansion technique.

When equations (17) are substituted into equations (16b), (16c), and (16d), and like powers of R_m are set equal to each other, the following equations are obtained.

$$\left.\begin{aligned} \frac{dB_{x_o}}{dy} &= 0 \\ \frac{\partial p_o}{\partial x} &= \eta \frac{d^2 u_o}{dy^2} \\ \frac{\partial p_o}{\partial y} &= 0 \\ \\ \frac{dB_{x_1}}{dy} &= -\frac{u_o B_o}{U_o \delta} \\ \frac{\partial p_1}{\partial x} + \frac{B_o^2}{\delta\mu U_o} u_o &= \eta \frac{d^2 u_1}{dy^2} \\ \frac{\partial p_1}{\partial y} &= 0 \\ \\ \frac{dB_{x_2}}{dy} &= -\frac{u_1 B_o}{U_o \delta} \\ \frac{\partial p_2}{\partial x} + \frac{B_o^2}{\delta\mu U_o} u_1 &= \eta \frac{d^2 u_2}{dy^2} \\ \frac{\partial p_2}{\partial y} &= -\frac{B_{x_1}}{\mu}\frac{dB_{x_1}}{dy} \\ \text{etc.} \end{aligned}\right\} \tag{18}$$

The derivatives of u have been written as ordinary derivatives because u is a function of y only. The successive functions are found by the Laplace transform method. As pointed out in the foregoing discussion on the first method, the first approximation is the nonmagnetic one. The results for the velocity, pressure gradient, and magnetic field terms for the series (17) are then found from equation (18) as

$$\left.\begin{aligned} B_{x_0} &= 0 \\ u_0 &= U_0\left(1 - \frac{y^2}{\delta^2}\right) \\ \frac{\partial p_0}{\partial x} &= -\frac{2\eta U_0}{\delta^2} \end{aligned}\right\} \tag{19a}$$

$$\left.\begin{aligned} B_{x_1} &= -\frac{B_0 y}{U_0 \delta}\left(1 - \frac{y^2}{3\delta^2}\right) \\ u_1 &= \frac{{B_0}^2}{\delta\eta\mu}\frac{y^2}{12}\left(1 - \frac{y^2}{\delta^2}\right) \\ \frac{\partial p_1}{\partial x} &= -\frac{5}{6}\frac{{B_0}^2}{\delta\mu} \end{aligned}\right\} \tag{19b}$$

$$\left.\begin{aligned} B_{x_2} &= -\frac{B_0}{\delta U_0}\frac{{B_0}^2}{\mu\delta\eta}\frac{y^3}{36}\left(1 - \frac{3y^2}{5\delta^2}\right) \\ u_2 &= \frac{1}{U_0 6!}\left(\frac{{B_0}^2}{\delta\eta\mu}\right)^2\left[-3y^2\delta^2 + y^4\left(5 - \frac{2y^2}{\delta^2}\right)\right] \\ \frac{\partial p_2}{\partial x} &= -\frac{\eta}{U_0}\left(\frac{{B_0}^2}{\delta\eta\mu}\right)^2\frac{\delta^2}{5!} \end{aligned}\right\} \tag{19c}$$

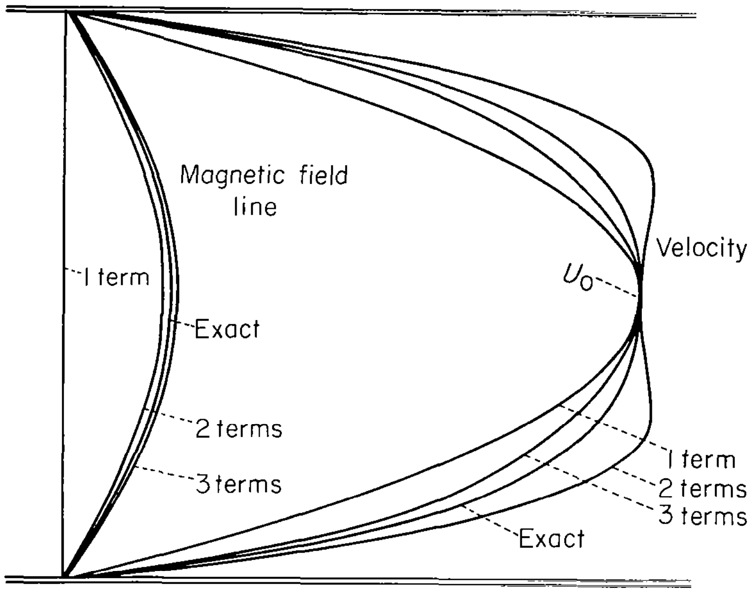
etc.

The relations (19) can be shown to be equivalent to the first few terms of a series expansion of the exact expressions (15). The velocity, for instance, can be written as

$$\frac{u}{U_0} = 1 - \frac{\cosh M(y/\delta) - 1}{\cosh M - 1}$$

$$= 1 - \frac{(M^2y^2/\delta^2 2!) + (M^4y^4/\delta^4 4!) + (M^6y^6/\delta^6 6!) + \ldots}{(M^2/2!) + (M^4/4!) + (M^6/6!) + \ldots}$$

As is to be expected, the expressions for the velocity in the set (19) agree precisely with the first three terms obtained when the fraction is expanded. The terms in the velocity and magnetic field expressions are then functions of only M when the quantities $B_o^2/\eta\mu$ and R_m in equations (17) and (19) are combined. The sum of the various approximations for the velocity and magnetic field is illustrated in sketch (b) for a magnetic Reynolds number R_m of 0.4 and a Hartmann parameter M of 4.



Sketch (b)

A value for R_m is needed for the displacement of the magnetic field lines. The rather large value of 4 for M was chosen to illustrate how the curves scatter about the exact one and is quite high for rapid convergence of the series. When M is 2, two terms of the series suffice. The exact shape of the magnetic field lines is obtained when equation (15c) is integrated with respect to y once, and the lines are assumed to be fixed to the wall.

Second method.- The differential equations (16), the series (17), and the boundary conditions of the previous part of this section apply here also. The difference between the two methods lies in the differential equation for the velocity. The form of equation (16c) is rewritten as

$$\frac{\partial p}{\partial x} + \sigma B_o^2 u = \eta \frac{\partial^2 u}{\partial y^2} \tag{20}$$

where the term $\sigma B_o^2 u$ is now assumed not to be negligible in the first approximation. When the series expressions (17) are inserted into equations (16b), (16d), and (20), and the terms containing the same powers of R_m are equated, the following set of differential equations is obtained

$$\left.\begin{aligned}
\frac{dB_{x_o}}{dy} &= 0 \\
\frac{\partial p_o}{\partial x} + \sigma B_o{}^2 u_o &= \eta \frac{d^2 u_o}{dy^2} \\
\frac{\partial p_o}{\partial y} &= 0 \\
\\
\frac{dB_{x_1}}{dy} &= -\frac{u_o B_o}{\delta U_o} \\
\frac{\partial p_1}{\partial x} + \sigma B_o{}^2 u_1 &= \eta \frac{d^2 u_1}{dy^2} \\
\frac{\partial p_1}{\partial y} &= 0 \\
\\
\frac{dB_{x_2}}{dy} &= -\frac{u_1 B_o}{\delta U_o} \\
\frac{\partial p_2}{\partial x} + \sigma B_o{}^2 u_2 &= \eta \frac{d^2 u_2}{dy^2} \\
\frac{\partial p_2}{\partial y} &= -\frac{B_{x_1}}{\mu}\frac{dB_{x_1}}{dy} \\
\text{etc.} &
\end{aligned}\right\} \qquad (21)$$

The solution to the set (21) is then found as

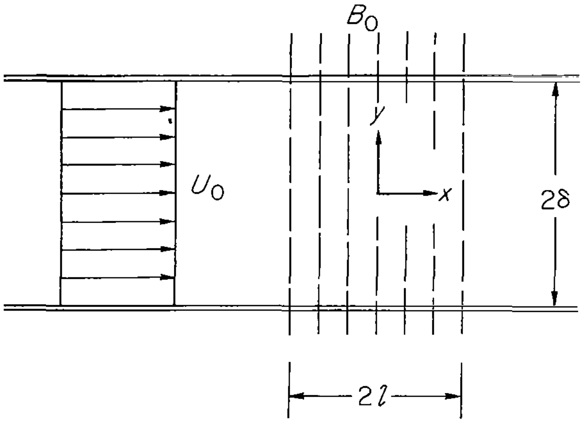
$$\left.\begin{aligned}
B_{x_o} &= 0 \\
u_o &= U_o \frac{\cosh M - \cosh M(y/\delta)}{\cosh M - 1} \\
p_o &= P - \frac{\eta U_o M^2}{\delta^2}(x - x_o)\frac{\cosh M}{1 - \cosh M} \\
B_{x_1} &= -B_o \frac{(y/\delta)\cosh M - (1/M)\sinh M(y/\delta)}{\cosh M - 1} \\
u_1 &= p_1 = 0 \\
p_2 &= \frac{B_o{}^2}{M^2}\left[\frac{y(M/\delta)\cosh M - \sinh M(y/\delta)}{\cosh M - 1}\right]^2 \\
u_2 &= u_3 = \ldots = B_{x_2} = B_{x_3} = \ldots = p_3 = p_4 = \ldots 0
\end{aligned}\right\} \qquad (22)$$

The higher order terms are all zero because the magnetic field component B_y is not changed by the fluid flow. The series terminates at the point where it has duplicated the exact solution (15). Once again the induced magnetic field B_x was computed by a simple integration of the equation $dB_{x_1}/dy = -u_0 B_0/\delta U_0$. The integral (12) could have been used instead but it is a longer process.

It is to be noted that the complete and exact expressions for the flow field are obtained very rapidly and much more easily by the second method than by the first method.

Inviscid Channel Flow With Finite Magnetic Field

Channel of finite height.- The magnetic field in the preceding example was assumed to extend far upstream and downstream from the observation point. The present analysis considers the steady-state flow disturbance caused by a magnetic field of constant strength and of finite extent in the x direction,[2] sketch (c). Once again the fluid is assumed to be flowing in an annulus so that the electric current lines are closed loops within the fluid and $\vec{E}$ may be set equal to zero. In order to simplify the analysis, the fluid will be assumed to be inviscid so that the velocity is constant across the channel in the first approximation. It will be seen that the higher order effects would be a great deal more difficult to find if the velocity profile were complicated in the slightest manner.



Sketch (c)

Since the initial velocity profile is not affected by viscosity and the magnetic field is uniform across the channel, the same velocity profile is computed by both methods in the first approximation. The difference lies in that the first method does not predict the pressure gradient $\partial p/\partial x$ caused by the fluid-magnetic field interaction until the second step whereas the second method predicts it in the first step. The second method then contains the first and second steps of the first method and will therefore be used.

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[2]The rectangular distribution of the magnetic field which was chosen to simplify the algebraic expressions is considered to be a first approximation to the shape of the magnetic field generated by an iron core magnet. A rectangular distribution could be approached by the use of an electric current sheet of large height at the upstream and downstream boundaries of the magnetic field. The presence of the wires or metal sheets carrying the electric current would add new phenomena onto the fluid motion.

The differential equations which describe the flow field are given by

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\sigma}{\rho} \left(uB_y^2 - vB_xB_y \right) = 0 \tag{23a}$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\sigma}{\rho} \left(uB_xB_y - vB_x^2 \right) = 0 \tag{23b}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{23c}$$

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0 \tag{23d}$$

$$\mu J_z = \frac{R_m}{U_o \delta} \left(uB_y - vB_x \right) = \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) \tag{23e}$$

The boundary conditions are

$$u = U_o \quad \text{at} \quad x = -\infty$$

$$B_x = 0 \quad \text{at} \quad y = 0$$

$$p = P \quad \text{at} \quad x = x_o \, , \; y = 0$$

$$v_o = 0 \quad \text{at} \quad y = \pm\delta$$

$$v = 0 \quad \text{at} \quad y = 0$$

$$B_{x_o} = 0$$

$$B_{y_o} = B_o \quad \text{at} \quad -l < x < l$$

The v velocity at $y = \pm\delta$ is assumed to be zero in the zeroth approximation only. It is necessary to assume flexible walls for the first iterated solution so that it can be found without an excessive amount of effort.

Once again the various functions are expanded in the series expressions

$$\left.\begin{aligned} u &= u_o + u_1 R_m + u_2 R_m^2 + \ldots \\ v &= v_o + v_1 R_m + v_2 R_m^2 + \ldots \\ B_x &= B_{x_o} + B_{x_1} R_m + B_{x_2} R_m^2 + \ldots \\ B_y &= B_{y_o} + B_{y_1} R_m + B_{y_2} R_m^2 + \ldots \\ p &= p_o + p_1 R_m + p_2 R_m^2 + \ldots \end{aligned}\right\} \tag{24}$$

When the equations (24) are inserted into the differential equations (23) and the terms containing the same power of R_m are equated, the equations for the first approximation are

$$\left.\begin{aligned} \frac{\partial B_{y_o}}{\partial y} &= 0 \\ \frac{\partial p_o}{\partial x} + \sigma u_o B_{y_o}^2 &= 0 \\ \frac{\partial p_o}{\partial y} &= 0 \\ \frac{\partial u_o}{\partial x} &= 0 \end{aligned}\right\} \tag{25}$$

The solution is

$$\left.\begin{aligned} B_{x_o} &= 0 \\ B_{y_o} &= B_o \\ u_o &= U_o \\ p_o - P &= -(x - x_o)\sigma U_o B_o^2 \qquad -l < x < l\ ; \quad -\delta < y < \delta \\ J_z &= \sigma U_o B_o \qquad -l < x < l\ ; \quad -\delta < y < \delta \end{aligned}\right\} \tag{26}$$

The electric current J_z which flows through the fluid is of uniform density in the rectangular area occupied by the magnetic field. It generates a magnetic field that causes the flow disturbances which are to be computed in the first iteration (subscript 1). The magnetic field components are found from equation (12) as

$$R_m B_{x_1} = \frac{\sigma\mu U_o B_o}{4\pi} \int_{-l}^{l} \int_{-\delta}^{\delta} \int_{-\infty}^{\infty} \frac{(y - y')dz'dy'dx'}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}} \tag{27a}$$

$$R_m B_{y_1} = -\frac{\sigma\mu U_o B_o}{4\pi} \int_{-l}^{l} \int_{-\delta}^{\delta} \int_{-\infty}^{\infty} \frac{(x - x')dz'dy'dx'}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}} \quad (27b)$$

When the integration has been carried out, the components of the induced magnetic field for the first iteration are

$$B_{x_1} = \frac{B_o}{4\pi\delta}\left[(x - l)ln \frac{(x - l)^2 + (y - \delta)^2}{(x - l)^2 + (y + \delta)^2} - (x + l)ln \frac{(x + l)^2 + (y - \delta)^2}{(x + l)^2 + (y + \delta)^2} + \right.$$

$$2(y - \delta)\tan^{-1}\frac{x - l}{y - \delta} + 2(y + \delta)\tan^{-1}\frac{x + l}{y + \delta} - 2(y - \delta)\tan^{-1}\frac{x + l}{y - \delta} -$$

$$\left. 2(y + \delta)\tan^{-1}\frac{x - l}{y + \delta}\right] \quad (28a)$$

$$B_{y_1} = -\frac{B_o}{4\pi\delta}\left[(y - \delta)ln \frac{(x - l)^2 + (y - \delta)^2}{(x + l)^2 + (y - \delta)^2} - (y + \delta)ln \frac{(x - l)^2 + (y + \delta)^2}{(x + l)^2 + (y + \delta)^2} + \right.$$

$$2(x - l)\tan^{-1}\frac{y - \delta}{x - l} - 2(x - l)\tan^{-1}\frac{y + \delta}{x - l} + 2(x + l)\tan^{-1}\frac{y + \delta}{x + l} -$$

$$\left. 2(x + l)\tan^{-1}\frac{y - \delta}{x + l}\right] \quad (28b)$$

From the equations (23) and (24), the differential equations for the fluid motion are

$$u_o \frac{\partial u_1}{\partial x} + \frac{1}{\rho}\frac{\partial p_1}{\partial x} + \frac{2\sigma B_{y_1} u_o B_o}{\rho} + \frac{\sigma B_o^2 u_1}{\rho} = 0 \quad (29a)$$

$$u_o \frac{\partial v_1}{\partial x} + \frac{1}{\rho}\frac{\partial p_1}{\partial y} - \frac{\sigma B_{x_1} u_o B_o}{\rho} = 0 \quad (29b)$$

$$\frac{\partial B_{x_1}}{\partial x} + \frac{\partial B_{y_1}}{\partial y} = 0 \quad (29c)$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0 \quad (29d)$$

$$\frac{B_o}{\delta} = \frac{\partial B_{y_1}}{\partial x} - \frac{\partial B_{x_1}}{\partial y} \quad (29e)$$

The pressure p_1 is eliminated from equations (29a) and (29b) by differentiating them with respect to y and x, respectively, and subtracting the two results. Since $u_o = U_o$, the equation becomes

$$U_o \frac{\partial^2 u_1}{\partial x\, \partial y} + \frac{\sigma U_o B_o}{\rho} \frac{\partial B_{y_1}}{\partial y} + \frac{\sigma B_o{}^2}{\rho} \frac{\partial u_1}{\partial y} - U_o \frac{\partial^2 v_1}{\partial x^2} = 0 \tag{30}$$

If the flow is assumed to be irrotational, the first and last terms cancel; that is, $\partial u_1/\partial y = \partial v_1/\partial x$, so that $U_o(\partial B_{y_1}/\partial y) + B_o(\partial u_{1p}/\partial y) = 0$. Therefore, the particular integral is simply

$$u_{1p} = - \frac{U_o}{B_o} B_{y_1} + f(x) \tag{31a}$$

Similarly,

$$v_{1p} = \frac{U_o}{B_o} B_{x_1} + g(y) \tag{31b}$$

The electric current equation together with the continuity equation furnishes the necessary expressions to solve for the functions $f(x)$ and $g(y)$. From the continuity equation it is found that

$$\frac{\partial u_{1p}}{\partial x} + \frac{\partial v_{1p}}{\partial y} = \frac{U_o}{B_o}\left(\frac{\partial B_{x_1}}{\partial y} - \frac{\partial B_{y_1}}{\partial x}\right) + \frac{\partial f(x)}{\partial x} + \frac{\partial g(y)}{\partial y} = 0$$

The electric current equation is used to eliminate the magnetic field components so that the functions are determined by

$$\frac{\partial f(x)}{\partial x} + \frac{\partial g(y)}{\partial y} = \left\{\begin{array}{ll} \dfrac{U_o}{\delta} & \text{inside magnetic field region} \\ 0 & \text{outside magnetic field region} \end{array}\right\} \tag{32}$$

Since either of the functions

$$f = \frac{xU_o}{\delta} + \text{constant} \qquad g = 0$$

or

$$f = 0 \qquad g = \frac{yU_o}{\delta} + \text{constant}$$

will satisfy equation (32), the solution is not unique unless a boundary condition is used to eliminate one or the other in the region of the imposed magnetic field. The functions f and g introduce a type of step function at $x = \pm l$. The arc tangent functions also have a step at the boundaries and could be thought to compensate for this discontinuity in both B_{x_1} and B_{y_1}. The relations $f = 0$ and $g = yU_o/\delta$ will be chosen arbitrarily and the arc tangents adjusted to yield smooth functions for u_1 and v_1.

A general solution which contains the complementary solution can be found by separation of variables after one of the velocity components has been eliminated from equation (30) by the continuity equation. The general solution is of such a form that it appears difficult to adapt it to suitable boundary conditions. Since the purpose of this paper is to illustrate effects and not to solve specific problems, the particular solution will be taken as the desired solution. The velocity components and the pressure gradients for the first iteration are then given by

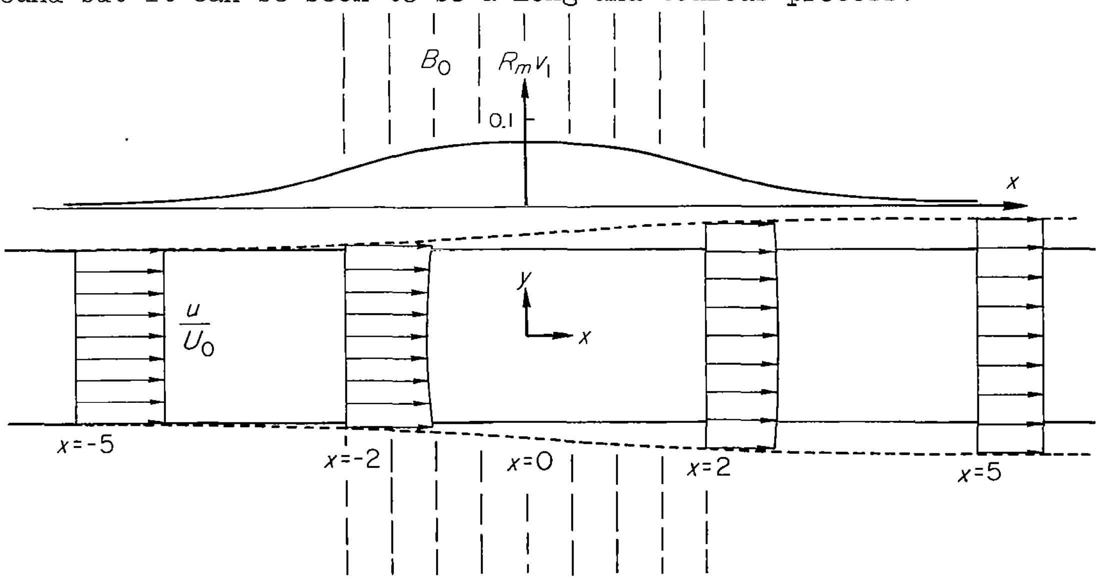
$$u_1 = -\frac{U_o}{B_o} By_1 \tag{33a}$$

$$v_1 = \frac{U_o}{B_o} Bx_1 + \frac{yU_o}{\delta} \tag{33b}$$

$$-\frac{\partial p_1}{\partial x} = \sigma B_o^2\left(u_1 + 2U_o \frac{By_1}{B_o}\right) - \rho \frac{U_o^2}{B_o} \frac{\partial By_1}{\partial x} \tag{33c}$$

$$\frac{\partial p_1}{\partial y} = \sigma Bx_1 B_o U_o - \rho \frac{U_o^2}{B_o} \frac{\partial Bx_1}{\partial x} \tag{33d}$$

The vertical velocity v_1 as given by equation (33b) at the planes $y = \pm\delta$ is not zero. In order that fluid is not required to flow through solid walls, the walls will be imagined to be made of an elastic substance which is deformed so that it corresponds with the stream surfaces. The flow is then an irrotational stream with the boundaries specified by the path taken by the rectangularly shaped core of fluid which enters at $x = -\infty$. If solid boundaries were placed at $y = \pm\delta$, the flow would be rotational. Typical streamline and velocity profile shapes computed by equations (26) and (33) are shown in sketch (d) for $\delta = 1$, $l = 2$, and a magnetic Reynolds number of 0.1. The second iteration will not be found but it can be seen to be a long and tedious process.



Sketch (d)

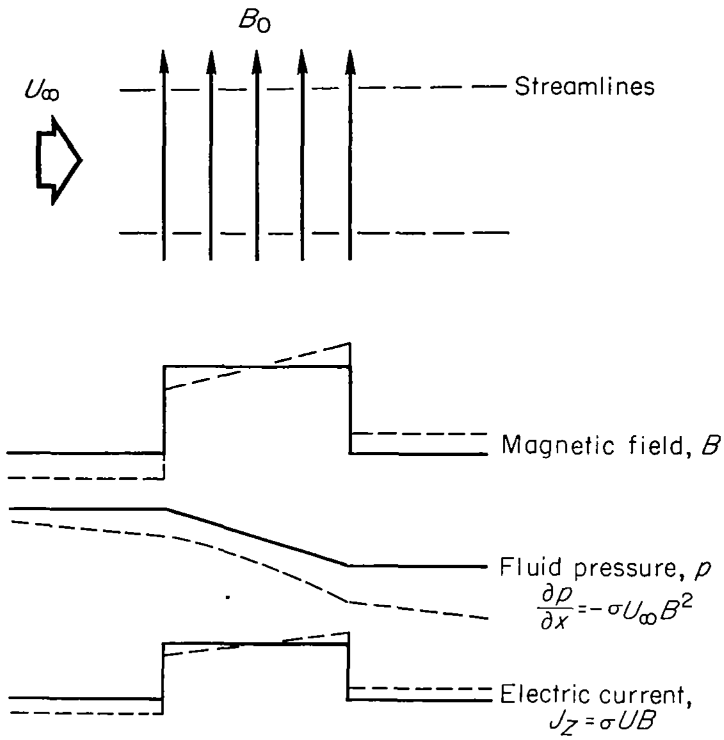
Channel of infinite height.- The expressions for the first iteration in the foregoing example are long in view of the simple geometry which was chosen. The length of equations (28) illustrates the complexity to be expected in the expressions for the magnetic field when an iteration on the basic flow field is attempted in other problems. The relations for the past example simplify a great deal if the width l of the magnetic field is allowed to become small while at the same time the field of view is restricted to the flow near the center of the channel. The ratio l/δ of the width l to the height δ of the channel becomes vanishingly small. The length of the expression for the induced magnetic field $\vec{B}_1$ is reduced because two of the limits are pushed out to infinity. The imposed magnetic field is again approximated by a rectangular distribution. The differential equations (23) and the boundary conditions which were used for the channel of finite height apply here also. The first approximation is once more represented by equation (26). The magnetic field components for the first iteration simplify to

$$\left.\begin{aligned}
B_{x_1} &= \frac{B_O}{4\pi\delta}\int_{-l}^{l}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(y-y')dz'dy'dx'}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}} = 0 \\
B_{y_1} &= -\frac{B_O}{4\pi\delta}\int_{-l}^{l}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(x-x')dz'dy'dx'}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}} = \begin{cases} 2B_O & x>l \\ 2xB_O & -l<x<l \\ -2B_O & x<-l \end{cases}
\end{aligned}\right\} \tag{34}$$

The velocity and pressure for the iterated flow field are then found to be

$$\left.\begin{aligned}
u_1 &= 0 \\
v_1 &= 0 \\
R_m\frac{\partial p_1}{\partial x} &= \begin{cases} -\sigma U_O R_m{}^2 B_O{}^2 & x < -l \\ -\sigma U_O B_O{}^2[(1+2R_m x)^2 - 1] & -l < x < l \\ -\sigma U_O B_O{}^2 R_m{}^2 & x > l \end{cases} \\
\frac{\partial p_1}{\partial y} &= 0
\end{aligned}\right\} \tag{35}$$

The variations of the flow parameters with streamwise distance as computed by equations (26) and (35) are shown schematically in sketch (e) for a magnetic Reynolds number of 0.25. The solid lines represent the first approximation given by equations (26) and the dashed lines represent the sum of the first approximation and the first iteration.



Sketch (e)

LARGE MAGNETIC REYNOLDS NUMBER EXAMPLES

A number of papers have been written describing the flow solutions obtained on the assumption that the magnetic Reynolds number is infinite and therefore that the magnetic lines of force are frozen into the fluid (see, e.g., the partial list represented by refs. 15 to 33). The method of approximation ($R_m = \infty$) lowers the order of the differential equation and leads to difficulties associated with the fact that the term containing the highest derivative of the differential equation was discarded. Problems of this type are sometimes called singular perturbation problems. The purpose of this section is to present two examples in which these difficulties associated with the equation for the magnetic field are illustrated. One additional example which is not affected by assuming that the magnetic Reynolds number is infinite for the first approximation is also presented. These cases may be briefly described as follows:

1. Hartmann type channel flow

2. Channel flow starting impulsively

3. Inviscid channel flow starting impulsively

The last part of this section discusses the circumstances under which a power-series expansion in time will exhibit the same singular perturbation characteristics brought about by the series in $1/R_m$.

Hartmann Type Channel Flow

The channel flow model first analyzed by Hartmann will be reworked by expanding the functions in a power series in $1/R_m$. Naturally, the sum of the series should yield the exact solution given by equations (15). The flow field geometry used in the small Reynolds number example will be used here; that is, the flow is assumed to be contained in an annulus which has a large diameter and length compared with its radial depth. The flow field is then treated as being two dimensional with the electric current lines as closed loops within the fluid and around the inner cylinder. All derivatives with respect to time are set equal to zero without consideration being given as to how the resulting flow was established. The differential equations which describe the present problem (sketch (a)) are then given by

$$\frac{\partial B_y}{\partial y} = 0 \tag{36a}$$

$$\frac{\partial B_x}{\partial y} = -\sigma\mu(E_z + uB_o) \tag{36b}$$

$$\frac{\partial p}{\partial x} = \frac{B_y}{\mu}\frac{\partial B_x}{\partial y} + \eta\frac{\partial^2 u}{\partial y^2} \tag{36c}$$

$$\frac{\partial p}{\partial y} = -\frac{B_x}{\mu}\frac{\partial B_x}{\partial y} \tag{36d}$$

with

$$\frac{\partial B_x}{\partial t} = -\frac{\partial E_z}{\partial y} = 0 \quad \text{and} \quad v = E_x = E_y = 0$$

The boundary conditions are

$$\begin{aligned} u &= 0 && \text{at} \quad y = \pm\delta \\ u &= U_o && \text{at} \quad y = 0 \\ B_x &= 0 && \text{at} \quad y = 0 \\ p &= P && \text{at} \quad x = x_o\,, \quad y = 0 \end{aligned}$$

The series expressions that are to be used to solve the set (36) consist of

$$u = u_o + \frac{u_1}{R_m} + \frac{u_2}{R_m^2} + \cdot \cdot \cdot \tag{37a}$$

$$v = v_o + \frac{v_1}{R_m} + \frac{v_2}{R_m^2} + \cdot \cdot \cdot \tag{37b}$$

$$p = p_o + \frac{p_1}{R_m} + \frac{p_2}{R_m^2} + \cdot \cdot \cdot \tag{37c}$$

$$B_x = B_{x_o} + \frac{B_{x_1}}{R_m} + \frac{B_{x_2}}{R_m^2} + \cdot \cdot \cdot \tag{37d}$$

$$B_y = B_{y_o} + \frac{B_{y_1}}{R_m} + \frac{B_{y_2}}{R_m^2} + \cdot \cdot \cdot \tag{37e}$$

$$E_z = E_{z_o} + \frac{E_{z_1}}{R_m} + \frac{E_{z_2}}{R_m^2} + \cdot \cdot \cdot \tag{37f}$$

When equations (37) are inserted into equations (36), and the terms containing the same powers of $1/R_m$ are equated, the following sets of equations are found.

$$\left.\begin{aligned} \frac{\partial B_{y_o}}{\partial y} &= 0 \\ E_{z_o} + u_o B_{y_o} &= 0 \\ \frac{\partial E_{z_o}}{\partial y} &= 0 \\ \frac{\partial p_o}{\partial x} &= \frac{B_{y_o}}{\mu}\frac{\partial B_{x_o}}{\partial y} + \eta\,\frac{\partial^2 u_o}{\partial y^2} \\ \frac{\partial p_o}{\partial y} &= -\,\frac{B_{x_o}}{\mu}\frac{\partial B_{x_o}}{\partial y} \end{aligned}\right\} \tag{38}$$

$$
\left.
\begin{aligned}
&\frac{\partial B_{y_1}}{\partial y} = 0 \\
&E_{z_1} + u_1 B_o = -U_o \delta \frac{\partial B_{x_o}}{\partial y} \\
&\frac{\partial E_{z_1}}{\partial y} = 0 \\
&\frac{\partial p_1}{\partial x} = \frac{B_{y_o}}{\mu} \frac{\partial B_{x_1}}{\partial y} + \frac{B_{y_1}}{\mu} \frac{\partial B_{x_o}}{\partial y} + \eta \frac{\partial^2 u_1}{\partial y^2} \\
&\frac{\partial p_1}{\partial y} = -\frac{B_{x_o}}{\mu} \frac{\partial B_{x_1}}{\partial y} - \frac{B_{x_1}}{\mu} \frac{\partial B_{x_o}}{\partial y}
\end{aligned}
\right\} \tag{39}
$$

etc.

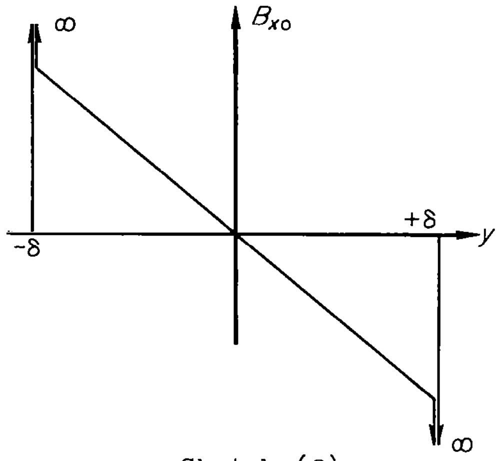
If the one-sided Laplace transform is applied to equations (38) and symmetry is imposed, the solution for the first ($R_m = \infty$) approximation is found as

$$
\left.
\begin{aligned}
&\bar{u}_o = U_o \frac{1 - e^{-\delta s}}{s} \\
&\bar{E}_{z_o} = -U_o B_o \frac{1 - e^{-\delta s}}{s} \\
&\bar{B}_{x_o} = \frac{\mu}{s^2 B_o} \frac{\partial p_o}{\partial x} + \eta\mu \frac{U_o}{B_o} e^{-\delta s} \\
&\bar{B}_{y_o} = \frac{B_o}{s}
\end{aligned}
\right\} \tag{40}
$$

where the bar over the letters designates the quantity transformed with respect to y. When the functions are inverted and it is remembered that the flow field is symmetrical about $y = 0$, the solution is

$$
\left.
\begin{aligned}
&B_{x_o} = \frac{y\mu}{B_o} \frac{\partial p_o}{\partial x} + \frac{\eta\mu U_o}{B_o} \Delta(|y| - \delta) \\
&B_{y_o} = 0 \\
&E_{z_o} = \begin{cases} -U_o B_o & -\delta < y < \delta \\ 0 & |y| > \delta \end{cases} \\
&u_o = \begin{cases} U_o & -\delta < y < \delta \\ 0 & y = \pm\delta \end{cases}
\end{aligned}
\right\} \tag{41}
$$

where $\Delta(|y| - \delta)$ is the Dirac delta function which has the character of being zero everywhere except at $|y| = \delta$ where it is infinite. The one-sided Laplace transform is being used so that negative values of y are not considered. The magnetic field variation is illustrated in sketch (f). The next and higher approximations are obtained by formal iteration by use of the relations (40) with care taken to retain the step functions and their derivatives.



Sketch (f)

$$\left.\begin{aligned} \bar{u}_1 &= -\frac{\delta\eta\mu s U_o}{B_o^2}\left(\frac{1}{s^2\eta}\frac{\partial p_o}{\partial x} + U_o e^{-\delta s}\right) \\ \bar{B}_{x_1} &= \frac{\delta\eta^2\mu^2 s^2 U_o}{B_o^3}\left(\frac{1}{s^2\eta}\frac{\partial p_o}{\partial x} + U_o e^{-\delta s}\right) \end{aligned}\right\} \tag{42a}$$

$$\left.\begin{aligned} \bar{u}_2 &= -\frac{\delta^2\eta^2\mu^2 s^3 U_o^2}{B_o^4}\left(\frac{1}{s^2\eta}\frac{\partial p_o}{\partial x} + U_o e^{-\delta s}\right) \\ \bar{B}_{x_2} &= \frac{\delta^2\eta^3\mu^3 s^4 U_o^2}{B_o^5}\left(\frac{1}{s^2\eta}\frac{\partial p_o}{\partial x} + U_o e^{-\delta s}\right) \end{aligned}\right\} \tag{42b}$$

etc.

The iterated quantities (42) possess a singularity at the wall which grows stronger with each successive step. The only change in the profiles with each higher order term occurs at the wall, and this change is represented by a singularity. With the exception of the first approximation, each term by itself is meaningless. However, the nature of the series can be recognized as an expansion of the fraction $1/1 - x = 1 + x + x^2 + x^3 + \ldots$; that is, the series

$$\bar{u} = \frac{U_o}{s} - \frac{1}{\eta s^3}\frac{\partial p_o}{\partial x}\left[\frac{s^2}{\sigma B_o^2/\eta} + \frac{s^4}{(\sigma B_o^2/\eta)^2} + \ldots\right] - \frac{U_o e^{-\delta s}}{s}\left(1 + \frac{s^2}{\sigma B_o^2/\eta} + \ldots\right)$$

$$\bar{B}_x = \frac{\mu\eta}{B_o}\left(\frac{1}{s^2\eta}\frac{\partial p_o}{\partial x} + U_o e^{-\delta s}\right)\left[1 + \frac{s^2}{\sigma B_o^2/\eta} + \left(\frac{s^2}{\sigma B_o^2/\eta}\right)^2 + \ldots\right]$$

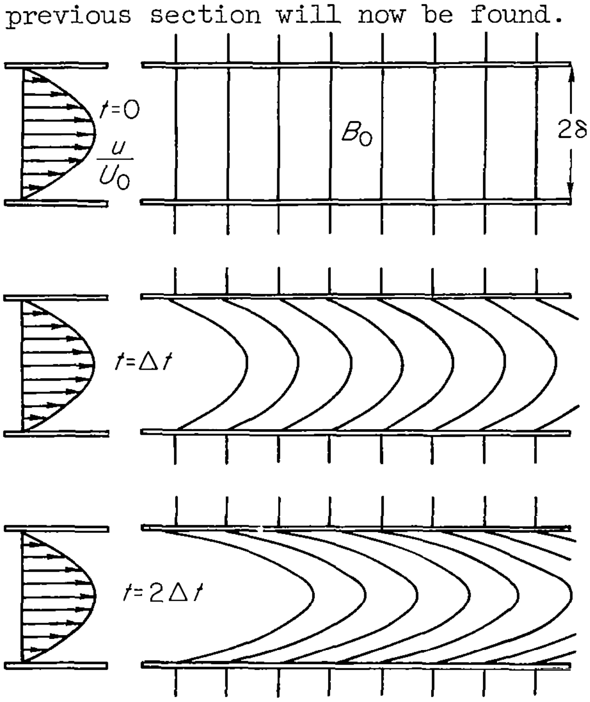
becomes

$$\left.\begin{aligned}\bar{u} &= \frac{U_0}{s}\left\{1 - \frac{e^{-\delta s}}{1-[s^2/(\sigma B_0{}^2/\eta)]}\right\} - \frac{1}{\eta}\frac{\partial p_0}{\partial x}\frac{1}{s(\sigma B_0{}^2/\eta)}\frac{1}{1-[s^2/(\sigma B_0{}^2/\eta)]}\\ \bar{B}_x &= \frac{\mu}{B_0}\left(\frac{1}{s^2}\frac{\partial p_0}{\partial x} + \eta U_0 e^{-\delta s}\right)\frac{1}{1-[s^2/(\sigma B_0{}^2/\eta)]}\end{aligned}\right\} \quad (43)$$

When equations (43) are inverted and the pressure gradient $\partial p_0/\partial x$ is adjusted so that $u = U_0$ at $y = 0$, the velocity, pressure, and magnetic field components are identical with the equations (15). It is to be noted that the series must be summed before it can be inverted because each term by itself yields an unrealistic result. It is to be expected that a straightforward formal expansion in many problems of this type will have this character. Since it will not generally be possible to sum the series to obtain the correct solution, the foregoing approach could be used only in special cases.

Channel Flow Starting Impulsively

A description of how the final ($t \to \infty$) velocity and magnetic field profiles are established in the channel flow problem treated in the previous section will now be found. In the beginning, assume that the electrically conducting fluid is flowing through the channel with a prescribed profile. At the time, $t \geq 0$, a uniform magnetic field is imposed across the channel as shown in the top diagram of sketch (g). Before going into the mathematical details of the analysis, the nature of the solution which is being sought will be discussed.



Sketch (g)

It has been pointed out that at the start, $t = 0$ the magnetic lines of force are stretched straight across the channel. If the fluid is a perfect conductor, $\sigma = \infty$, the magnetic lines are effectively frozen into the material and convected downstream with it as illustrated in sketch (g). The magnetic Reynolds number (or conductivity) may be large but never really infinite. The stretching of the magnetic field lines cannot then go on indefinitely, but at some time a relative motion or slip between the lines and the fluid must start.

Equilibrium is established when the lines of force slip at the fluid velocity and are stationary relative to the walls. The equations describing the physical quantities at large time should be those found by Hartmann and repeated here as equations (15).

The equation of motion of the magnetic field may be written as

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{U} \times \vec{B}) + \frac{1}{\sigma\mu} \nabla^2 \vec{B}$$

If the term on the right is zero, the magnetic field is convected with the fluid. In the general case then, when this term is small but not zero, it represents the relative motion (or slip) between the magnetic lines of force and the fluid. The equation of motion for the magnetic field for the channel flow problem being considered may then be written as

$$\frac{\partial B_x}{\partial t} = B_o \frac{\partial u}{\partial y} + \frac{1}{\sigma\mu} \frac{\partial^2 B_x}{\partial y^2} \tag{44a}$$

or

$$= B_o \frac{\partial u}{\partial y} - B_o \frac{\partial u_s}{\partial y} \tag{44b}$$

where u_s is the slip velocity of the magnetic field lines through the fluid. Since the slope of a line of force is given by

$$\frac{B_x}{B_y} = \frac{B_x}{B_o} = \frac{dx}{dy} \tag{45}$$

the slip velocity may be written as

$$u_s = - \frac{1}{\sigma\mu B_o} \frac{\partial B_x}{\partial y} = - \frac{1}{\sigma\mu} \frac{d^2 x}{dy^2} \approx \text{Curvature of magnetic lines} \tag{46}$$

The physical significance of the statement (46) is seen to be plausible by consideration of an analogy which approximates the present model. It consists of the impulsive start of the flow of a thick substance like grease or heavy syrup through a two-dimensional channel with rubber bands stretched across it. The grease represents the electrically conducting fluid and the rubber bands represent the magnetic lines of force. Before the fluid starts to move the bands are straight across the channel or undistorted. When the grease begins to move, the rubber bands are displaced and stretched. In the initial stages, the two move together, but very shortly the rubber bands become taut enough to begin to cut through the grease and try to slip backward to their neutral position.

Counteracting this is the motion of the fluid which attempts to displace the bands a greater amount. Eventually, the rate at which the bands are slipping because of their tension equals the velocity of the grease, and equilibrium is reached. The ultimate displacement of the bands is proportional to the velocity U_0 of the grease, the length of the bands (or width of the channel), and inversely proportional to the solidity of the grease (or the rate of diffusion corresponding to $1/\sigma\mu$). The product of these quantities is the same as the magnetic Reynolds number $\sigma\mu U l$. Also, if the pump is turned off, the rubber bands will drift or sift back to their original or neutral position at a rate proportional to their curvature (see eq. (46)) much as a magnetic field would decay in the analogous system. The rate of decay would be similar to the exponential decay, $e^{-(t/\sigma\mu\delta^2)}$, of magnetic fields. If a sharp corner is artificially generated in the rubber bands, by a knife edge, for example, it would be expected to be smoothed out very quickly, thus once more bearing out the diffusive character of equation (46).

Attention is now turned from the physical to the mathematical part of the analysis. The differential equations for the flow in a two-dimensional channel when the magnetic Reynolds number is large which are to be used in conjunction with the set (37) are

$$\left.\begin{aligned}
\frac{\partial B_x}{\partial t} &= B_y \frac{\partial u}{\partial y} + \frac{1}{\sigma\mu}\frac{\partial^2 B_x}{\partial y^2} \\
E_z + uB_o &= -\frac{1}{\sigma\mu}\frac{\partial B_x}{\partial y} \\
\frac{\partial B_y}{\partial y} &= 0 \\
\frac{\partial E_z}{\partial y} &= -\frac{\partial B_x}{\partial t} \\
\rho\frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} &= \frac{B_y}{\mu}\frac{\partial B_x}{\partial y} + \eta\frac{\partial^2 u}{\partial y^2} \\
\frac{\partial p}{\partial y} &= -\frac{B_x}{\mu}\frac{\partial B_x}{\partial y}
\end{aligned}\right\} \qquad (47)$$

In the absence of a magnetic field the velocity profile is known to be parabolic as a result of viscous effects. It is found that the steady-state profile is also a parabola when a magnetic field is present if $R_m = \infty$ and the initial or $t = 0$ velocity distribution is not a constant. A parabolic velocity distribution will therefore be chosen as the initial profile in this example. The boundary conditions are then

$$u = 0 \qquad \text{at} \quad y = \pm\delta$$

$$u = U_o \qquad \text{at} \quad y = 0$$

$$B_x = 0 \qquad \text{at} \quad y = 0$$

$$u = U_o\left(1 - \frac{y^2}{\delta^2}\right) \qquad \text{at} \quad t = 0$$

$$B_x = 0 \qquad \text{at} \quad t = 0$$

$$E_z = 0 \qquad \text{for} \quad |y| \geq \delta$$

If the flow were inviscid the velocity at the wall would still be zero in the $R_m = \infty$ approximation so that the magnetic lines of force are not sheared off by the fluid motion or so that infinitely large magnetic pressures at the walls are avoided. The following sets of differential equations are obtained when the series (37) are substituted into equation (47) and the terms containing the same powers of $1/R_m$ are set equal to each other.

$$\frac{\partial B_{x_o}}{\partial t} = B_{y_o} \frac{\partial u}{\partial y} \tag{48a}$$

$$\frac{\partial B_{y_o}}{\partial y} = 0 \tag{48b}$$

$$E_{z_o} + u_o B_o = 0 \tag{48c}$$

$$\rho \frac{\partial u_o}{\partial t} + \frac{\partial p_o}{\partial x} = \frac{B_{y_o}}{\mu} \frac{\partial B_{x_o}}{\partial y} + \eta \frac{\partial^2 u_o}{\partial y^2} \tag{48d}$$

$$\frac{\partial p_o}{\partial y} = - \frac{B_{x_o}}{\mu} \frac{\partial B_{x_o}}{\partial y} \tag{48e}$$

$$\frac{\partial B_{x_1}}{\partial t} = B_{y_1}\frac{\partial u_o}{\partial y} + B_{y_o}\frac{\partial u_1}{\partial y} + U_o\delta\frac{\partial^2 B_{x_o}}{\partial y^2} \tag{49a}$$

$$\frac{\partial B_{y_1}}{\partial y} = 0 \tag{49b}$$

$$E_{z_1} + u_1 B_o = -U_o\delta\frac{\partial B_{x_o}}{\partial y} \tag{49c}$$

$$\rho\frac{\partial u_1}{\partial t} + \frac{\partial p_1}{\partial x} = \frac{B_{y_o}}{\mu}\frac{\partial B_{x_1}}{\partial y} + \frac{B_{y_1}}{\mu}\frac{\partial B_{x_o}}{\partial y} + \eta\frac{\partial^2 u_1}{\partial y^2} \tag{49d}$$

$$\frac{\partial p_1}{\partial y} = -\frac{B_{x_1}}{\mu}\frac{\partial B_{x_o}}{\partial y} - \frac{B_{x_o}}{\mu}\frac{\partial B_{x_1}}{\partial y} \tag{49e}$$

etc.

The solution to the set (48) is found by use of the Laplace transform with respect to time to reduce the partial differential equations to ordinary differential equations. The flow quantities when $R_m = \infty$ are then found as

$$B_{x_o} = -\frac{2U_oB_oyt}{\delta^2} \tag{50a}$$

$$B_{y_o} = B_o \tag{50b}$$

$$u_o = U_o\left(1 - \frac{y^2}{\delta^2}\right) \tag{50c}$$

$$p - P = -\left[\left(\frac{U_oB_o{}^2t}{\mu\delta^2} + \frac{\eta U_o}{\delta^2}\right)x + \frac{2U_o{}^2B_o{}^2y^2t^2}{\mu\delta^4}\right] \tag{50d}$$

The velocity profile did not change from the one imposed by the initial conditions because it represents the final profile. If another initial shape had been assumed, the velocity profile would change over to the parabolic form, equation (50c), as time progresses; that is, provided that the shape is not the rectangular distribution $u = U_o$ for $-\delta < y < \delta$ and with $u = 0$ at $y = \pm\delta$, which is also a stable shape. It is also interesting to note that the magnetic field does not take on a final profile as time increases indefinitely, but that it continues to grow linearly with time. (The distortion of the lines of force are illustrated in sketch (g).) It might be thought that the velocity and magnetic field configurations should approach the steady-state values given by equations (41) as time becomes large. Such does not appear to be the case. The example in the next section illustrates a set of circumstances which brings about the transition to the profiles (41).

Equations (49) are used in the next step of the iteration. The magnetic field component B_{y_1} is found to be zero from equation (49b) and the fact that $B_y = B_o$ outside of the channel. The equations (49d) and (49e) are combined to eliminate the pressure. The Laplace transform of that result and of equation (49c) with respect to time yields

$$\rho s \frac{d\bar{u}_1}{dy} = \frac{B_o}{\mu} \frac{d^2\bar{B}_{x_1}}{dy^2} + \eta \frac{d^3\bar{u}_1}{dy^3} \tag{51a}$$

$$s\bar{B}_{x_1} = B_o \frac{d\bar{u}_1}{dy} + U_o\delta \frac{d^2B_{x_o}}{dy^2} \tag{51b}$$

where the bar over the symbol again denotes the transformed quantity. The variables $\bar{B}_{x_1}$ and $\bar{u}_1$ are unknowns, whereas $d^2\bar{B}_{x_o}/dy^2$ is to be found from the first approximation. A difficulty now arises because $\bar{B}_{x_o}$ is a function of y to the first power only, and therefore the term $d^2\bar{B}_{x_o}/dy^2$ is zero, indicating no change in the velocity profile. The differential equation for $\bar{u}_1$ is then found as

$$\frac{d^3\bar{u}_1}{dy^3} - \frac{s^2}{\nu s + (B_o^2/\rho\mu)} \frac{d\bar{u}_1}{dy} = 0 \tag{52}$$

The iterated velocity u_1 is zero at the boundaries and at the center of the channel. Since equation (52) is homogeneous, the only solution for $\bar{u}_1$ is zero throughout the flow field. The same result is found for B_{x_1}, p_1, and for all the higher iterations. In other words, the higher order terms cannot be found by the formal iteration scheme being studied here. It is not known whether this is caused by the fact that only whole powers of R_m are considered or whether some other difficulty is responsible.

Inviscid Channel Flow Starting Impulsively

Iterated quantities can be found for the previous example if the fluid is assumed to have zero viscosity. In effect then one singular perturbation problem ($R_m = \infty$) is removed by introducing a new one ($Re = Ul/\nu = \infty$) by discarding the viscous term in the equation of motion for the fluid. The system of equations denoted as equations (48) and (49) is unchanged except that now $\eta = 0$. The result for the first approximation is given by equations (50) because it satisfies the differential equations with $\eta = 0$, and the boundary conditions are not altered. The velocity at the wall must be zero when $R_m = \infty$ so that the magnetic lines of force are not sheared off by the fluid motion. For the higher approximations, the fluid can slip past the lines and therefore need not have zero velocity at the wall. The boundary conditions on u_1 are that it be zero at $y = 0$ and that $\partial u/\partial y = 0$ at $y = 0$ or that the profile be symmetrical. Information regarding the previous step is carried forward by the relation (49c). The boundary conditions on B_{x_1} and on p_1 are unchanged. The solution for $\bar{u}_1$ is then

$$\bar{u}_1 = \frac{2U_o^2}{s^2\delta}\,\frac{1 - \cosh\, sy(\sqrt{\rho\mu/B_o^2})}{1 - \cosh\, s\delta(\sqrt{\rho\mu/B_o^2})} \tag{53}$$

which may be inverted by contour integration to yield

$$u_1 = \frac{2U_o^2}{\delta}\left(\frac{y^2}{\delta^2}\,t + \sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}\left\{\left[\frac{\delta}{\pi n(\sqrt{B_o^2/\rho\mu})}\,\sin\frac{2\pi nt}{\delta(\sqrt{\rho\mu/B_o^2})} - t\cos\frac{2\pi nt}{\delta(\sqrt{\rho\mu/B_o^2})}\right]\left(\cos\frac{2\pi ny}{\delta} - 1\right) + \frac{y}{\sqrt{B_o^2/\rho\mu}}\,\sin\frac{2\pi nt}{\delta(\sqrt{B_o^2/\rho\mu})}\,\sin\frac{2\pi ny}{\delta}\right\}\right) \tag{54}$$

A similar set of expressions is obtained for B_{x_1} and p_1. The form of equation (54) is simplified if the time is assumed large. The quantity s in equation (53) is then small and only the leading terms need be retained. The velocity u_1 at large time is then given by

$$\left.u_1\right)_{t\to\infty} = \frac{2U_o^2y^2t}{\delta^3} \tag{55a}$$

Similarly,

$$\left.B_{x_1}\right)_{t\to\infty} = \frac{2U_o^2B_oyt^2}{\delta^3} \tag{55b}$$

Without going into the tedious details of the analysis the second iteration yields

$$\left.u_2\right)_{t\to\infty} = -\frac{2U_o^3y^2t^2}{\delta^4} \tag{56a}$$

$$\left.B_{x_2}\right)_{t\to\infty} = -\frac{4}{3}\,\frac{U_o^3yB_ot^3}{\delta^4} \tag{56b}$$

If the process is continued, the general term in the series is recognized and the series summed; that is,

$$\left.u\right)_{t\to\infty} = U_o\left\{1 - \frac{y^2}{\delta^2}\sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{-2(tU_o/\delta)}{R_m}\right]^n\right\} \tag{57a}$$

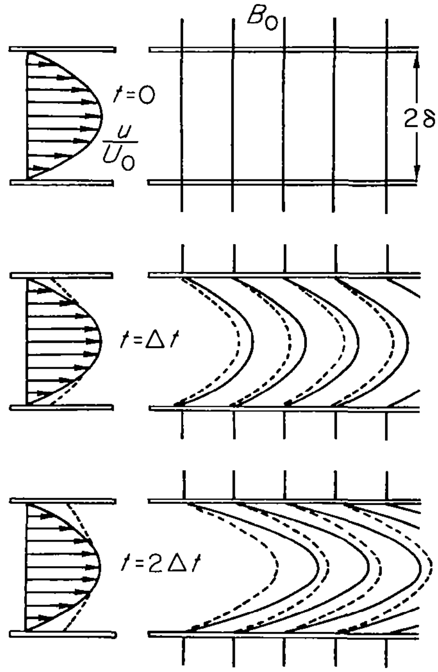
$$B_x)_{t\to\infty} = \frac{y}{\delta} B_o R_m \sum_{n=1}^{\infty} \frac{1}{n!} \left[\frac{-2(tU_o/\delta)}{R_m}\right]^n \tag{57b}$$

or

$$u)_{t\to\infty} = U_o\left(1 - \frac{y^2}{\delta^2} e^{-2tU_o/\delta R_m}\right) \tag{58a}$$

$$B_x)_{t\to\infty} = -\frac{y}{\delta} B_o R_m\left(1 - e^{-2tU_o/\delta R_m}\right) \tag{58b}$$

The relations (58) not only exhibit the correct characteristics at large time but also satisfy the initial ($t = 0$) conditions. They are not the general solution, however, as can be shown by inserting them into the differential equations (47). Since equations (58) exhibit the correct trends at both large and small time, they would probably serve as a fair estimate of what goes on in-between these limits. The velocity profile and position of the magnetic field lines as computed by equations (58) are shown as dashed lines in sketch (h) for $R_m = 5$. The solid lines are the corresponding lines for $R_m = \infty$.



Sketch (h)

Time-Magnetic Reynolds Number Equivalence

The flow in a two-dimensional channel is characterized at large time by the parameter $2tU_o/\delta R_m$ which reduces to $2t/\sigma\mu\delta^2$. The time history of a given flow configuration is then related to another by the inverse ratios of the conductivity. A parameter involving the ratio of time to conductivity appears sound on the basis of physical arguments and is often used as the time constant for the decay of magnetic fields. In particular, Elsasser (refs. 35 and 36) used it to analyze the magnetic field of the earth.

Flow fields which have their magnetic fields distorted from a given initial state may be treated by assuming that the magnetic Reynolds number is large or by expanding in a power series in time about the initial state. If the initial state is one in which the entire magnetic field is imposed by a system outside the flow field, no electric currents are flowing in the fluid and the quantity $\nabla^2\vec{B}$ is zero as a first approximation. Both of the series expansions ($1/R_m$ and time) are then of the singular perturbation type and difficulty is to be expected when higher approximations are obtained by iteration on the basic flow field.

The two illustrations of solutions found by a series expansion in $1/R_m$ could be worked by a series expansion in time. The result with viscosity is the same in both cases. Higher approximations for the series in time are not found in the case without viscosity unless the term $\rho(\partial u/\partial t)$ is discarded. Such a procedure could be thought to correspond to consideration of the velocity when the time becomes large. The expressions (57) and (58) then describe the results for large time obtained by either a power series in time or in inverse powers of the magnetic Reynolds number.

CONCLUDING REMARKS

The solutions of magnetohydrodynamic flow problems may be found by expansion of the various flow parameters in positive or negative powers of the magnetic Reynolds number $R_m = \sigma\mu U l$. The first technique $\left(U = \sum_n U_n R_m^n \text{ series}\right)$ is straightforward and appears to have no hidden mathematical difficulties so that successive approximations lead to the correct complete solution. It soon becomes obvious, however, that the integral (12), which is used to find the induced magnetic field, causes the expressions to become very long and impractically cumbersome to deal with. It is very tedious then to treat all but the very simplest of flow fields by this technique. It will generally be necessary to turn to numerical methods with electronic computers to obtain the higher order terms.

The examples which were worked out by expansion in a power series of inverse powers of the magnetic Reynolds number brought out some difficulties which are to be encountered. The fact that the highest derivative in the equation of motion for the magnetic field is discarded to obtain the first approximation leads to a singular perturbation type of problem when higher order solutions are sought. The higher order terms found by formal iteration on the basic $(R_m = \infty)$ solution possess characteristics which are unreal and therefore do not describe the physical problem.

In many problems relating to the growth or decay of magnetic fields, the solution will depend on a parameter of the form $t/\sigma\mu\delta^2$. The term in a series expansion in time will then resemble the terms in a series expansion in $1/R_m$. The series expansion in time may also be of the singular perturbation type if the term $\nabla^2\vec{B}$ is zero at the beginning of the sequence of events.

Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., Feb. 27, 1959

APPENDIX

ELECTROMAGNETIC PARAMETERS

Induced electric intensity

$$E = BU = (B,\ \text{lines/in.}^2)(U,\ \text{in./sec})10^{-8}\ \text{v/in.}$$

Force density

$$F = BJ = (B,\ \text{lines/in.}^2)(J,\ \text{amp/in.}^2)8.85\times10^{-8}\ \text{lb/in.}^3$$

Hartmann parameter

$$M = \sqrt{\frac{\sigma}{\eta}}\,Bl = (B,\ \text{lines/in.}^2)(l,\ \text{in.})\left(\sqrt{\frac{\sigma,\ \text{mhos/in.}}{\eta,\ \text{slugs/ft sec}}}\right)3.57\times10^{-5}$$

Electromagnetic parameter

$$Q = \frac{\sigma B^2 l}{\rho U} = \frac{(\sigma,\ \text{mhos/in.})(B,\ \text{lines/in.}^2)^2(l,\ \text{in.})}{(\rho/\rho_o)(U,\ \text{ft/sec})}\ 6.43\times10^{-10}$$

$$\rho_o = 0.002378\ \text{slugs/ft}^3$$

Magnetic Reynolds number

$$R_m = \sigma\mu U l = (\sigma,\ \text{mhos/in.})\left(\frac{\mu}{\mu_o}\right)(U,\ \text{ft/sec})(l,\ \text{in.})3.830\times10^{-7}$$

Magnetic pressure

$$p_B = \frac{B^2}{2\mu} = \frac{(B,\ \text{lines/in.}^2)^2}{\mu/\mu_o}\ 1.386\times10^{-8}\ \text{lb/in.}^2$$

Magnetic pressure number

$$R_h = \frac{B^2}{\rho\mu U^2} = \frac{(B,\ \text{lines/in.}^2)^2}{(\mu/\mu_o)(\rho/\rho_o)(U,\ \text{ft/sec})^2}\ 1.68\times10^{-3}$$

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